QUANTIZATION OF GEOMETRIC CLASSICAL r-MATRICES

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In this note we define geometric classical r-matrices and quantum R-matrices, and show how any geometric classical r-matrix can be quantized to a geometric quantum R-matrix. This is one of the simplest nontrivial examples of quantization of solutions of the classical Yang-Baxter equation, which can be explicitly computed. The idea of the above quantization was inspired by the results in [ESS]. We note that a construction similar to ours was obtained in [KLM].

1 Geometric classical r-matrices and quantum R-matrices

Let X be a smooth, affine algebraic variety over $\mathbb{C}$.

Definition 1 *A geometric classical r-matrix on X is a derivation $r : \mathbb{C}[X \times X] \to \mathbb{C}[X \times X]$ (i.e. a vector field on $X \times X$), which satisfies the classical Yang-Baxter equation*

$$[r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}] = 0 \text{ in } \mathbb{C}[X \times X \times X] \tag{1.1}$$

and the unitarity condition

$$r + r_{21} = 0 \text{ in } \mathbb{C}[X \times X]. \tag{1.2}$$

Example 1. Let X be any variety as above, and v a vector field on X. Define $r^v(x,y) = (v(x), -v(y))$. Then r is a geometric classical r-matrix. We call it a permutation r-matrix, since it corresponds to an "infinitesimal permutation" of X given by v.

Example 2. Let X be a finite dimensional algebra over $\mathbb{C}$ (e.g. a matrix algebra), and the vector field r be given by $r_c(x,y) = (xcy, -ycx)$, where $c \in X$. It can be checked that r_c is a geometric classical r-matrix.

Definition 2 *A formal diffeomorphism g of a smooth affine variety Y is an algebra homomorphism $g : \mathbb{C}[Y] \to \mathbb{C}[Y][[\hbar]]$ such that $g = 1 + O(\hbar)$.*

In particular, if v is a vector field on Y, then one can define a formal diffeomorphism $g = e^{\hbar v}$ of Y by $(gF)(x) = \sum_{m \geq 0} \frac{\hbar^m v^m}{m!} F(x)$. The last expression can be written as $F(e^{\hbar v}x)$, where $e^{\hbar v}x$ is a regular map from the formal disk to Y.

Definition 3 *A geometric quantum R-matrix on X is a formal diffeomorphism of $X \times X$, which satisfies the quantum Yang-Baxter equation*

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12} \tag{1.3}$$

and the unitarity condition

$$RR_{21} = 1. \tag{1.4}$$

This definition is a modification of Drinfeld's definition of a (unitary) set-theoretical solution of the quantum Yang-Baxter equation (see [Dr]), in the case when X is an algebraic variety. The term "geometric" is used because the map $R : \mathbb{C}[X^2] \to \mathbb{C}[X^2][[\hbar]]$ is not an arbitrary linear map, but a map of geometric origin, i.e. coming from a formal diffeomorphism of X^2.

Example 3. Let X be as in Example 1. For any formal diffeomorphism g, define $R^g(x,y) = (g(x), g^{-1}(y))$. This is a geometric quantum R-matrix. We call it a permutation R-matrix, since it corresponds to a "formal permutation" of X given by g.

Example 4. Let X be a finite dimensional algebra over $\mathbb{C}$, and the formal diffeomorphism R be given by $R_c(x,y) = (x(1+\hbar cy), y(1+\hbar cx+\hbar^2 cxcy)^{-1})$, where $c \in X$. It was checked in [ESS] (see formula (A5)) that R is a geometric quantum R-matrix.

Suppose that R is a geometric quantum R-matrix on X, and its $\hbar$-expansion looks like $R = 1 + \hbar r + O(\hbar^2)$. Then it is easy to check that r is a geometric classical r-matrix.

Definition 4 *R is said to be a quantization of r.*

Example 5. It is easy to see that R^g is a quantization of r^v if $g = e^{\hbar v}$, and R_c is a quantization of r_c.

Our main result is the following quantization theorem.

Theorem 1.1 *Any geometric classical r-matrix admits a quantization.*

Remark. It was proved in [EK] that any classical r-matrix can be quantized. In the unitary case, this was proved earlier by Drinfeld. However, these results don't automatically guarantee that if the r-matrix r is geometric then it has a quantization which is also geometric. So the main theorem does not obviously follow from the general quantization results. Also, the main theorem has the advantage that its proof gives an easy way to compute the quantization.

2 Proof of the main theorem

2.1. The Lie algebra with a bijective 1-cocycle associated to a geometric classical r-matrix.

Let r be a geometric classical r-matrix on X. Then r is a vector field on X^2, so it is an element of $\mathrm{Vect}X \otimes \mathbb{C}[X] \oplus \mathbb{C}[X] \otimes \mathrm{Vect}(X)$, where $\mathrm{Vect}(X)$ is the Lie algebra of vector fields on X. Consider the space $\mathfrak{g} = \{(1 \otimes f)(r) | f \in (\mathrm{Vect}(X))^* \oplus (\mathbb{C}[X])^*\}$. It follows from (1.1) – (1.2) that $\mathfrak{g}$ is a finite dimensional Lie subalgebra in the Lie algebra $\mathrm{Vect}(X) \ltimes \mathbb{C}[X]$ of differential operators of order ≤ 1 on X. Moreover, $\mathfrak{g} = \mathfrak{g}_+ \oplus \mathfrak{g}_-$ as a vector space, where $\mathfrak{g}_+ = \{(1 \otimes f)(r) | f \in (\mathbb{C}[X])^*\}$, $\mathfrak{g}_- = \{(f \otimes 1)(r) | f \in (\mathrm{Vect}[X])^*\}$ are Lie subalgebras, and $r \in \mathfrak{g}_+ \otimes \mathfrak{g}_- \oplus \mathfrak{g}_- \otimes \mathfrak{g}_+$. Since $[\mathfrak{g}_+, \mathfrak{g}_-] \subset \mathfrak{g}_-$, the space $V = \mathfrak{g}_-$ has a $\mathfrak{g}_+$-module structure. We introduce a bijective map $\phi_r : V^* \to \mathfrak{g}_+$ by the formula $\phi_r(f) = (1 \otimes f)(r)$ (here f is extended by 0 to $\mathfrak{g}_+$). Denote $\pi = \phi_r^{-1} : \mathfrak{g}_+ \to V^*$.

Lemma 1 *$\pi : \mathfrak{g}_+ \to V^*$ is a bijective 1-cocycle. That is, for any $a, b \in \mathfrak{g}_+$, $\pi([a, b]) = a * \pi(b) - b * \pi(a)$, where $*$ denotes the $\mathfrak{g}_+$ action on V^*.*

Proof of the Lemma.

Let $f, g \in V^*$, $x \in \mathfrak{g}_+^*$, then

$$\begin{aligned}[\phi_r(f), \phi_r(g)](x) &= (x \otimes f \otimes g)([r_{12}, r_{13}]) \\ &= -(x \otimes f \otimes g)([r_{12}, r_{23}] + [r_{13}, r_{23}]) \\ &= -f([(x \otimes 1)(r), (1 \otimes g)(r)]) + g([(x \otimes 1)(r), (1 \otimes f)(r)]) \\ &= -(\phi_r(g) * f)(x) + (\phi_r(f) * g)(x),\end{aligned}$$

which proves the Lemma. □

2.2. Exponentiation of the bijective cocycle.

Recall some basic facts about formal groups. Let L be any Lie algebra over $\mathbb{C}$. We denote by $E(L)$ the group of formal expressions of the form $e^{\hbar b}$, where $b \in L[[\hbar]]$, which are multiplied by the Campbell-Hausdorff formula. This is the group of $\mathbb{C}[[\hbar]]$-rational points of the formal group associated to the Lie algebra L.

It is clear that E is a functor from the category of Lie algebras to the category of groups. That is, to any homomorphism $\phi : L \to L'$ of Lie algebras there corresponds a homomorphism of groups $E(\phi) : E(L) \to E(L')$.

Let $\psi : L \to Vect(X)$ be a homomorphism of Lie algebras. Then the formal group $E(L)$ acts on X *on the right* by formal diffeomeorphisms, via $e^{hb} \to e^{h\psi(b)}$. We stress that this is a right, and not left, action, i.e. the above assignment is an antihomomorphism, rather than homomorphism, of groups.

Now let us come back to the proof of the theorem. Define a linear map $\bar{\pi} : \mathfrak{g}_+ \to \mathfrak{g}_+ \ltimes V^*$ by $\bar{\pi}(a) = (a, \pi(a))$. Lemma 1 states that $\bar{\pi}$ is a homomorphism of Lie algebras.

Let $G_+ = E(\mathfrak{g}_+)$. Since $\mathfrak{g}_+$ is a Lie algebra of vector fields on X, the group G_+ acts on X *on the right* by formal diffeomorphisms. We denote this action by ρ, i.e. $F(\rho(e^{hb})x) = e^{hb}F(x)$ for $b \in \mathfrak{g}_+,\ F \in \mathbb{C}[X]$.

Also, it is clear that G_+ acts naturally on $V^*[[\hbar]]$. Consider the group $G_+ \ltimes V^*[[\hbar]]$ with multiplication $(a, b)(a', b') = (aa', b + a * b')$. It is easy to see that this group is naturally isomorphic to $E(\mathfrak{g}_+ \ltimes V^*)$ (here $\ltimes$ is the semidirect product). Therefore, the Lie algebra homomorphism $\bar{\pi}$ can be lifted to a group homomorphism $\bar{\Pi} : G_+ \to G_+ \ltimes V^*[[\hbar]]$. Let $p : G_+ \ltimes V^*[[\hbar]] \to V^*[[\hbar]]$ be the projection map. The bijective map $\Pi : G_+ \to V^*[[\hbar]]$ defined as a composition $\Pi = \hbar^{-1} p \bar{\Pi}$ satisfies the 1-cocycle relation $\Pi(aa') = a * \Pi(a') + \Pi(a)$.[1]

Let $\varepsilon : X \to \mathbb{C}[X]^*$ be the evaluation map. Restriction of its values to V gives a map $\tilde{\varepsilon} : X \to V^*$.

For $x, y \in X$ we define $x \circ y = \rho(\Pi^{-1}(-\tilde{\varepsilon}(x)))y$ (the right hand side is a map of the formal disk to X). Define a homomorphism of algebras $R : \mathbb{C}[X \times X] \to \mathbb{C}[X \times X][[\hbar]]$ by the formula

$$(RF)(x, y) = F((y\circ)^{-1}x, ((y\circ)^{-1}x) \circ y), \tag{2.1}$$

where $(y\circ)^{-1}$ denotes the inverse operator to the action of y by $\circ$. It is easy to see that $R = 1 + O(\hbar)$.

Now we will use the following result from [ESS], Section 2.4.

[1]Note that this definition of a 1-cocycle differs from the one used in [ESS] by the transformation $\Pi(a) \to \Pi(a^{-1})$.

Proposition. Let G be a group acting on a set X *on the right* via a map $\rho : G \to Aut(X)$. Let A be an abelian group with *a left* G-action, and $\pi : G \to A$ a bijective 1-cocycle. Let $\phi : X \to A$ be a G-antiinvariant map, i.e for any $g \in G,\ x \in X\ g\phi(x) = \phi(\rho(g^{-1})x)$. Define $R : X \times X \to X \times X$ by

$$R(x,y) = ((y\circ)^{-1}x, ((y\circ)^{-1}x) \circ y), \tag{2.2}$$

where $x \circ y = \rho(\pi^{-1}(\phi(x)))y$. Then R satisfies the unitarity condition and the quantum Yang-Baxter equation.

Remark. [ESS] dealt with a left action of the group G on X. The above proposition is a reformulation of the corresponding result from Section 2.4 of [ESS] in terms of the right G-action on X.

This proposition (or, more precisely, its version for formal groups) implies that R is a geometric quantum R-matrix.

It is easy to compute directly that $R = 1 + \hbar r + O(\hbar^2)$. Thus, R is a quantization of r. The theorem is proved. $\square$

3 Example

Let us show that for Examples 1 and 2, the procedure of the previous section gives the same quantizations as in Examples 3 and 4.

For Example 1, this is clear: the Lie algebra $\mathfrak{g}_+$ is 1-dimensional, and the computation is trivial. So let us consider Example 2.

We have: X is a finite dimensional algebra, and $r_c(x,y) = (xcy, -ycx)$. In this case it is easy to check that $\mathfrak{g}_+$ is the right ideal cX generated by c, with commutator given by $[a,b] = ab - ba$. The representation V^* of $\mathfrak{g}_+$ is $\mathfrak{g}_+$ itself, with $a * b = -ba$. The bijective 1-cocycle π has the form $\pi(a) = a$. Let us exponentiate the cocycle π. We have $\bar{\pi}(a) = (a,a)$, and

$$\bar{\Pi}(e^{\hbar a}) = e^{\hbar(a,a)} = (e^{\hbar a}, \frac{e^{\hbar a} - 1}{a} * a) = (e^{\hbar a}, 1 - e^{-\hbar a}).$$

Thus, $\Pi(A) = \hbar^{-1}(1 - A^{-1})$.

The map $\tilde{\varepsilon}$ has the form $\tilde{\varepsilon}(x) = cx$. Thus, $x \circ y = \rho(\Pi^{-1}(-cx))y = \rho((1+\hbar cx)^{-1})y = y(1+\hbar cx)^{-1}$. Therefore, we get

$$(RF)(x,y) = F(x(1+\hbar cy), y(1+\hbar cx + \hbar^2 cxcy)^{-1}),$$

which coincides with the geometric quantum R-matrix of Example 4.

4 Geometric classical r-matrices on the line and their quantization

Theorem 4.1 *Let r be a geometric classical r-matrix on the affine line, which is not a permutation r-matrix. Then r reduces to $r(n) = xy^n\frac{\partial}{\partial x} - yx^n\frac{\partial}{\partial y}$ for some $n \geq 1$, after a linear change of variables.*

Proof Let $\mathfrak{g}_+$ be the Lie algebra in $\mathbb{C}[x]\frac{\partial}{\partial x}$ associated to r. The $\mathfrak{g}_+$-module $\mathbb{C}[x]$ has a nonzero finite dimensional submodule V. If r is not a permutation matrix, this submodule cannot consist only of constants. This implies that $\mathfrak{g}_+$ is a Lie subalgebra of the 2-dimensional Lie algebra spanned by $\partial/\partial x$ and $x\partial/\partial x$ (if $\mathfrak{g}_+$ has an element $(x^i + ..)\partial/\partial x$ with $i > 1$ then it generates an infinite dimensional space from any nonconstant polynomial). Consider two cases.

1. $\mathfrak{g}_+$ is 2-dimensional. Then $V =< 1, x >$, and one can check that V^* is isomorphic to the adjoint representation of $\mathfrak{g}_+$. So if r with such $\mathfrak{g}_+$ exists, then there must exist a nondegenerate 1-cocycle from $\mathfrak{g}_+$ to $\mathfrak{g}_+$, i.e. a nondegenerate derivation of $\mathfrak{g}_+$. It is easy to check that such a derivation does not exist, so this case is impossible.

2. $\mathfrak{g}_+$ is 1-dimensional. Then it is spanned by an element $(ax + b)\partial/\partial x$, such that $a \neq 0$. After a change of variable we can assume that $\mathfrak{g}_+$ is spanned by $x\partial/\partial x$. Then $V = \mathfrak{g}_-$ has to be spanned by x^n for some $n \geq 1$, so $r = c(xy^n\frac{\partial}{\partial x} - yx^n\frac{\partial}{\partial y})$.

This proves the proposition, as c can be easily scaled out. $\square$

Now let us discuss quantization of $r(n)$. If $n = 1$, this was done above: $r(1) = r_c$ from Example 2, where $X = \mathbb{C}$, and $c = 1$. So the quantization given by the procedure of Section 2 is $R(1)(x, y) = (x(1 + \hbar y), \frac{y}{1+\hbar x+\hbar^2 xy})$.

If $n > 1$, let us make a change of variables $x \to x^n, y \to y^n$. This change maps $r(n)$ to $r(1)/n$. Applying the same change of variables to the quantum R-matrix, we get the following quantization of $r(n)$.

$$R(n)(x, y) = (x(1 + n\hbar y^n)^{1/n}, y(1 + n\hbar x^n + n^2\hbar^2 x^n y^n)^{-1/n}).$$

It is easy to check that the procedure of Section 2 gives the same answer.

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